AH National Qualifications 2016

X702/77/11

Mathematics of Mechanics

TUESDAY, 17 MAY

1:00 PM – 4:00 PM

Total marks — 100

Attempt ALL questions.

You may use a calculator.

Full credit will be given only to solutions which contain appropriate working.

State the units for your answer where appropriate. Any rounded answer should be accurate to three significant figures (or one decimal place for angles in degrees) unless otherwise stated.

Write your answers clearly in the answer booklet provided. In the answer booklet, you must clearly identify the question number you are attempting.

Use **blue** or **black** ink.

Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not you may lose all the marks for this paper.

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A/HTP

FORMULAE LIST

Newton's inverse square law of gravitation

$$F = \frac{GMm}{r^2}$$

Simple harmonic motion

$$v^2 = \omega^2(a^2 - x^2)$$

$$x = a\sin(\omega t + \alpha)$$

Centre of mass

Triangle: $\frac{2}{3}$ along median from vertex.

Semicircle: $\frac{4r}{3\pi}$ along the axis of symmetry from the diameter.

Standard derivatives	
$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Standard integrals	
$f(x)$	$\int f(x)\,dx$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax) + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
e^{ax}	$\frac{1}{a}e^{ax} + c$

Total marks — 100

Attempt ALL questions

Candidates should observe that $g\,\mathrm{m\,s^{-2}}$ denotes the magnitude of the acceleration due to gravity. Where appropriate, take its magnitude to be $9{\cdot}8\,\mathrm{m\,s^{-2}}$.

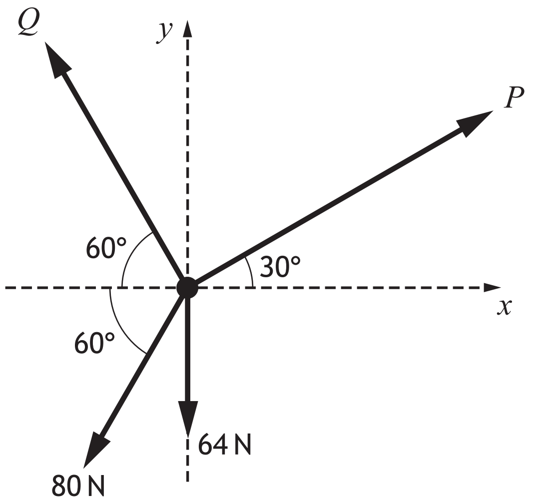
1. A bicycle and rider have a total mass of 70 kg. They are travelling at $12\,\mathrm{m\,s^{-1}}$. The cyclist applies the brakes for 1·5 seconds, resulting in a total resistive force of 180 newtons.

What is the speed of the bicycle after 1·5 seconds? **3**

2. In a children's playground game, four light inextensible ropes are attached at one end to a small toy ring.

Four children each take the other end of a rope and pull it taut.

The ring is in equilibrium and the whole system is in a horizontal plane with appropriate axes as shown in the diagram.



The tensions in the four ropes are P, Q, 80 and 64 newtons respectively, and their directions relative to the axes are shown.

Calculate the magnitude of the tensions P and Q. **4**

3. A constant force $\mathbf{F} = (2\mathbf{i} + 3\mathbf{j})$ N acts on a particle as it moves in a straight line from point A to point B with position vectors $(-3\mathbf{i} + \mathbf{j})$ metres and $(6\mathbf{i} + 4\mathbf{j})$ metres respectively.

Calculate the work done by the force. **3**

[Turn over

MARKS

4. Find the equation of the tangent to the curve $y = x \ln x$ at the point where $x = e$. **3**

5. The tip of a saw oscillates with simple harmonic motion.

- When the tip is 5 mm from its centre of motion it has a velocity of $2\ \text{m s}^{-1}$.
- When it is 7 mm from the centre it has a velocity of $1\ \text{m s}^{-1}$.

Calculate the amplitude of the motion and find the number of oscillations in one second. **5**

6. A remote controlled aircraft is flown from point A to point B. It accelerates for 10 seconds at a constant rate from rest to a take-off speed of $15\ \text{m s}^{-1}$.

Once airborne, it accelerates for a further 20 seconds at a slower constant rate to a cruising speed of $u\ \text{m s}^{-1}$.

It maintains this speed for 60 seconds until it lands.

The aircraft then decelerates for 10 seconds to a complete stop.

(a) Sketch a speed-time graph of the journey, clearly showing all the important information. **2**

(b) (i) If the distance travelled from A to B is 1·725 km, calculate the value of u. **2**

(ii) State one assumption you have made about the path of the aircraft during your calculations. **1**

7. An object of mass 9 kg starts from rest at an origin and moves in a straight line so that its acceleration in m s^{-2} is given as $a = 4 - \sqrt{t}$, where t is the time in seconds.

Calculate its maximum speed and hence the increase in kinetic energy. **4**

8. (a) Show that $\dfrac{3x^3 + 8x^2 - 11}{(x+1)(x+3)(x-2)}$ can be written as $3 + \dfrac{2x^2 + 15x + 7}{x^3 + 2x^2 - 5x - 6}$. **3**

(b) Hence express $\dfrac{3x^3 + 8x^2 - 11}{(x+1)(x+3)(x-2)}$ in partial fractions. **4**

MARKS

9. A velodrome has a circular track of radius 30 metres, banked at an angle of 32° to the horizontal. The coefficient of friction between a bicycle tyre and the track is 0·3.

(a) Once the cyclist reaches maximum speed without the bicycle slipping, he cycles for 5 minutes. Assuming he maintains this speed, how many full laps does he complete? **6**

(b) Given that air resistance can be ignored and the cyclist is treated as a particle, what other assumption has been made? **1**

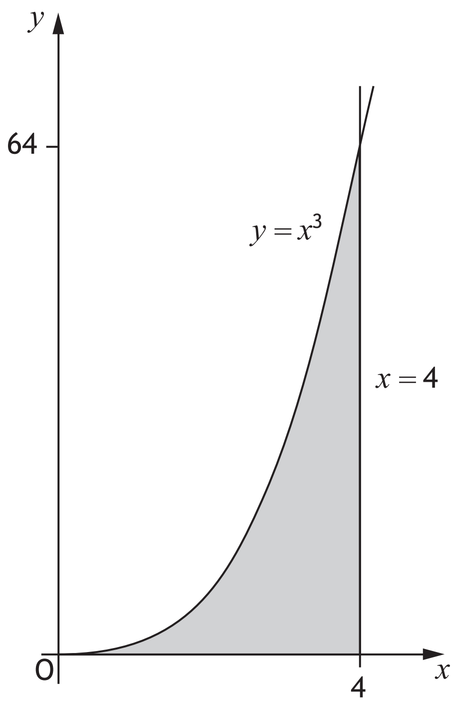
10. A stone is thrown from the top of a cliff and the subsequent motion can be modelled in the xy plane by the equations $x = 4t$ **and** $y = 20 + 2t - 5t^2$.

(a) Use parametric differentiation to find $\frac{dy}{dx}$ in terms of t. **2**

(b) (i) Find the angle of projection of the stone. **2**

(ii) By considering $\frac{dy}{dx}$ find the value of t when the stone is moving at 45° below the horizontal. **2**

11. A uniform lamina is bounded by the curve $y = x^3$, the line $x = 4$ and the x-axis.



Find the coordinates of the centre of mass of the lamina. **4**

[Turn over

12. An aircraft flies 1080 km due east from Glasgow to Copenhagen in a time of $2\frac{1}{4}$ hours.

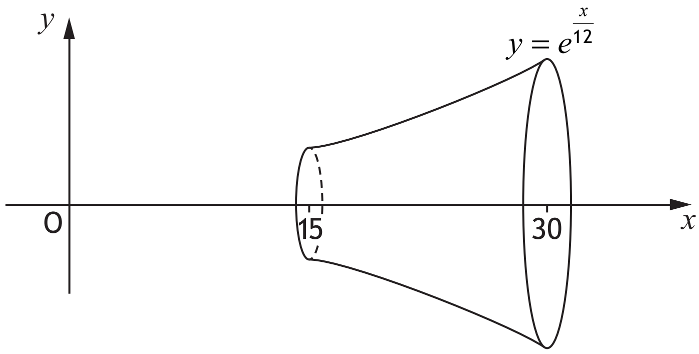
The aircraft sets a course on a bearing of 100° and the speed of the aircraft in still air is $450\ \text{km h}^{-1}$.

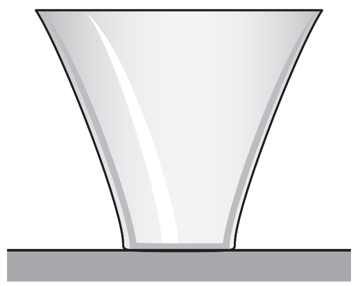
(a) Calculate the magnitude and direction of the wind. **3**

(b) (i) Given that the velocity of the wind remains constant, explain why the return journey will take longer. **1**

(ii) Calculate how much longer the return journey will take, giving your answer to the nearest minute. **4**

13. A glass bowl is modelled by rotating the curve $y = e^{\frac{x}{12}}$ between $x = 15$ and $x = 30$ through 2π radians about the x-axis as shown in the diagram.





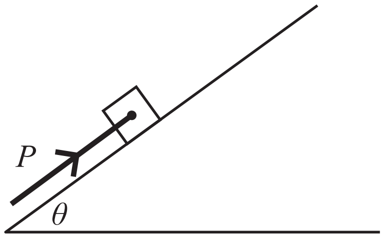
(a) Find the volume of the bowl. **3**

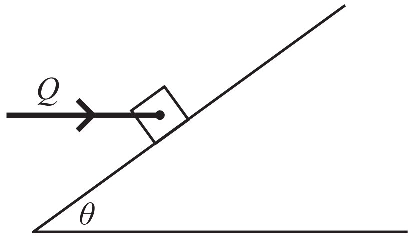
(b) A line is to be put on the bowl to indicate when it is half full.

How far above the base of the bowl should this line be marked? **3**

14. A block of weight W is placed on a rough inclined plane at an angle θ to the plane.

It can be held on the point of slipping down the plane by a force P acting parallel to the plane or a horizontal force Q as shown by the diagrams.





Prove that $P = \dfrac{QW}{Q\sin\theta + W\cos\theta}$. **7**

15. A mass of 0·25 kg is attached to a horizontal spring of natural length 1 metre and modulus of elasticity 20 newtons. The spring is stretched and then released. It experiences a resistive force of magnitude $6v$ newtons, where v is the velocity of the mass.

(a) Show that the subsequent motion satisfies the second order differential equation

$$\frac{d^2x}{dt^2} + 24\frac{dx}{dt} + 80x = 0.$$ **2**

(b) Solve this second order differential equation given that the mass is released from rest with an extension in the spring of 0·2 m. **6**

(c) Show that the acceleration is equal to zero when $t = \frac{1}{16}\ln 5$ seconds and find the displacement at this time. **3**

[Turn over for next question

16. A ball is projected from an origin on horizontal ground with speed $V\,\mathrm{m\,s^{-1}}$ at an angle of elevation of θ and moves freely under gravity. It passes through a point which is x metres horizontally from the origin at a height y metres above the ground.

(a) Show that the trajectory of the particle has equation

$$y = x\tan\theta - \frac{gx^2}{2V^2}\left(1+\tan^2\theta\right).$$

(Note that $\sec^2\theta = 1+\tan^2\theta$) **3**

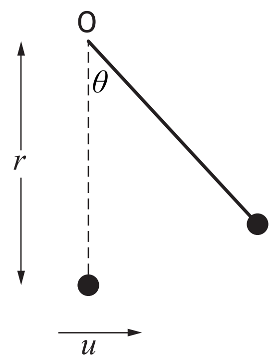
(b) The ball is at a vertical height of h metres when it has travelled $4h$ metres horizontally.

It is again at a height of h metres when it has travelled a further h metres horizontally.

Determine the angle of projection θ. **5**

17. A light inextensible string of length r metres has one end attached to a fixed point O and the other end is attached to a particle of mass m kilograms.

From its equilibrium position, the particle is given a horizontal velocity $u\,\mathrm{m\,s^{-1}}$, as shown in the diagram.



(a) (i) Show that the tension, T, in the string can be expressed as

$$T = \frac{mu^2}{r} + mg\left(3\cos\theta - 2\right)$$

where θ is the angle between the string and the downward vertical through O. **4**

(ii) Determine a condition for u in terms of r and g, so that the particle executes a complete circle. **2**

(b) Given that the value of u is $2\sqrt{rg}$, find an expression in terms of r for the height of the particle above its starting position when the string goes slack. **3**

[END OF QUESTION PAPER]